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N-soliton solutions to a $(2+1)$-dimensional integrable equation

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Abstract. We give explicitly N-soliton solutions of a $(2+1)$-dimensional equation, $\phi_{xt} + \phi_{xxxz}/4 + \phi_x\phi_{xz} + \phi_{xx}\phi_z/2 + \partial_x^{-1}\phi_{zzz}/4 = 0$. This equation is obtained by unifying two directional generalizations of the potential KdV equation: the closed ring with the potential KP equation, and the Calogero–Bogoyavlenskij–Schiff equation. This equation is also a reduction of the KP hierarchy. We also find the Miura transformation which yields the same ring of the corresponding modified equations.

The study of higher-dimensional integrable systems is one of the central themes in integrable systems. A typical example of a higher-dimensional integrable system is obtained by modifying the Lax operators of a basic equation, the potential KdV (p-KdV) equation in this paper. The Lax pair of the p-KdV equation have the form

$$L(x,t) = \partial_x^2 + \phi_x(x,t) \tag{1}$$

$$T(x,t) = (L(x,t)^{\frac{3}{2}})_+ + \partial_t = \partial_x L(x,t) + T'(x,t) + \partial_t \tag{2}$$

where $(\ )_+$ means the part of $(\ )$ with non-negative powers of ∂_x. The p-KdV equation

$$\phi_{xt} + \tfrac{1}{4}\phi_{xxxx} + \tfrac{3}{2}\phi_x\phi_{xx} = 0 \tag{3}$$

is equivalent to the Lax equation

$$[L, T] = 0. \tag{4}$$

Zakharov and Shabat modified the L operator to include another spatial dimension, showing that the inverse scattering method is applicable to it [1]. Using this method, Konopelchenko and Dubrovsky explicitly derived several $(2+1)$-dimensional integrable equations [2]. On the other hand, Calogero derived n-dimensional integrable equations using Wronskian integral relations [3–6]. This equation reduces to the potential Calogero–Bogoyavlenskij–Schiff (p-CBS) equation,

$$\phi_{xt} + \tfrac{1}{4}\phi_{xxxz} + \phi_x\phi_{xz} + \tfrac{1}{2}\phi_{xx}\phi_z = 0. \tag{5}$$

Bogoyavlenskij modified the T operator and gave the Lax pair of the above equation [7], whereas Schiff obtained the same equation by the reduction of the self-dual Yang–Mills equation [8]. The p-CBS equation (5) has N-soliton solutions [9]. We modified both the L and T operators searching for the $(3+1)$-dimensional integrable equation. However, the Lax equation was eventually reduced to a $(2+1)$-dimensional equation,

$$\phi_{xt} + \tfrac{1}{4}\phi_{xxxz} + \phi_x\phi_{xz} + \tfrac{1}{2}\phi_{xx}\phi_z + \tfrac{1}{4}\partial_x^{-1}\phi_{zzz} = 0. \tag{6}$$

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Equation (6) was found from a different view point in [7, 10]. This equation is a reduction of the KP hierarchy [11]. This equation admits the Lax representation [7, 12],

$$L = \partial_x^2 + \phi_x + \partial_z \tag{7}$$

$$T = \partial_x^2\partial_z + \tfrac{1}{2}\partial_z^2 + \tfrac{1}{2}\phi_z\partial_x + \phi_x\partial_z + \tfrac{3}{4}\phi_{xz} - \tfrac{1}{4}\partial_x^{-1}\phi_{zz} + \partial_t \tag{8}$$

and has the Painlevé property [12] in the sense of the Weiss–Tabor–Carnevale method [13].

In this paper we will give explicitly N-soliton solutions to this $(2+1)$-dimensional equation. We will also give the modified equation of (6) from the Miura transformation and its Lax pair. Moreover, we will clarify the relations among the equations obtained by extending both L and T and their modified equation.

By the dependent variable transformation

$$\phi \equiv 2\frac{\tau_x}{\tau} \tag{9}$$

equation (6) is transformed into the trilinear form

$$(36\mathcal{T}_x^2\mathcal{T}_t + \mathcal{T}_x^4\mathcal{T}_z^* + 8\mathcal{T}_x^3\mathcal{T}_x^*\mathcal{T}_z + 9\mathcal{T}_z^3)\tau\cdot\tau\cdot\tau = 0. \tag{10}$$

The operators $\mathcal{T}$, $\mathcal{T}^*$ are defined by [10, 14]

$$\mathcal{T}_z^n f(z)\cdot g(z)\cdot h(z) \equiv (\partial_{z_1} + j\partial_{z_2} + j^2\partial_{z_3})^n f(z_1)g(z_2)h(z_3)|_{z_1=z_2=z_3=z} \tag{11}$$

where j is the cubic root of unity, $j = \exp(2\mathrm{i}\pi/3)$. $\mathcal{T}_z^*$ is the complex conjugate operator of $\mathcal{T}_z$ obtained by replacing $(\partial_{z_1} + j\partial_{z_2} + j^2\partial_{z_3})$ by $(\partial_{z_1} + j^2\partial_{z_2} + j\partial_{z_3})$. Equation (10) was obtained by Hietarinta, Grammaticos and Ramani from the singularity analysis of the trilinear equation [10]. The one-soliton solution of equation (10) takes the form

$$\tau_1 = 1 + \exp(P_1x + Q_1z + R_1t + S_1) \tag{12}$$

and the dispersion relation is

$$P_1^4Q_1 + Q_1^3 + 4P_1^2R_1 = 0. \tag{13}$$

Here S_1 is a constant. Next, the form of the two-soliton solution is written as

$$\begin{aligned}\tau_2 = 1 &+ \exp(P_1x + Q_1z + R_1t + S_1) + \exp(P_2x + Q_2z + R_2t + S_2)\\ &+A_{12}\exp((P_1+P_2)x + (Q_1+Q_2)z + (R_1+R_2)t + S_1 + S_2)\end{aligned} \tag{14}$$

and A_{12} is

$$A_{12} = \frac{P_1^2P_2^2(P_1-P_2)^2 - (P_1Q_2 - P_2Q_1)^2}{P_1^2P_2^2(P_1+P_2)^2 - (P_1Q_2 - P_2Q_1)^2}. \tag{15}$$

As a result, we have the conjecture that N-soliton solutions of equation (10) have the form

$$\tau_N = 1 + \sum_{n=1}^{N}\sum_{{}_NC_n} A_{i_1\cdots i_n}\exp(\eta_{i_1} + \cdots + \eta_{i_n}) \tag{16}$$

$$\eta_j = P_jx + Q_jz + R_jt + S_j \qquad P_j^4Q_j + Q_j^3 + 4P_j^2R_j = 0 \tag{17}$$

$$A_{jk} = \frac{P_j^2P_k^2(P_j-P_k)^2 - (P_jQ_k - P_kQ_j)^2}{P_j^2P_k^2(P_j+P_k)^2 - (P_jQ_k - P_kQ_j)^2} \tag{18}$$

$$A_{i_1\ldots i_n} = A_{i_1,i_2}\ldots A_{i_1,i_n}\ldots A_{i_{n-1},i_n}. \tag{19}$$

Here the summation ${}_NC_n$ indicates summation over all possible combinations of n elements taken from N, and symbols S_j always denote arbitrary constants. However, this equation also allows the following the Wronskian-type solution (21), which is easier for the analytic proof of the solution than the form (16). We introduce new parameters p_j and q_j as follows,

$$p_j - q_j = P_j \qquad p_j^2 - q_j^2 = \pm Q_j \qquad p_j^4 - q_j^4 = \mp 2R_j. \tag{20}$$

We rewrite the N-soliton solution τ_N (16) in the form of an $N \times N$ Wronskian

$$\tau_N = \det\begin{pmatrix} f_1 & \cdots & \partial_x^{N-1} f_1 \\ \vdots & \dots & \vdots \\ f_N & \cdots & \partial_x^{N-1} f_N \end{pmatrix} \tag{21}$$

where

$$f_j = \exp(p_j x \pm p_j^2 z \mp \tfrac{1}{2} p_j^4 t + c_j) + \exp(q_j x \pm q_j^2 z \mp \tfrac{1}{2} q_j^4 t + d_j) \tag{22}$$

with constants, c_j and d_j.

Proof. We prove analytically that the Wronskian form (21) is indeed the solution to a trilinear p-CBS equation (10). For later use, we represent the Wronskian solution (21) as follows,

$$\tau_N = \det\begin{pmatrix} f_1 & \cdots & \partial_x^{N-1} f_1 \\ \vdots & \dots & \vdots \\ f_N & \cdots & \partial_x^{N-1} f_N \end{pmatrix} \equiv [0, \dots, N-1] \tag{23}$$

where the symbol j in $[\cdots]$ denotes the jth derivative of the column vector ${}^t(f_1, \dots, f_N)$. Then the derivatives of τ_N are described as

$$\tau_{N,x} = [0, \dots, N-2, N] \tag{24}$$

$$\tau_{N,z} = \mp[0, \dots, N-3, N-1, N] \pm [0, \dots, N-2, N+1] \tag{25}$$

$$\begin{aligned} \tau_{N,t} = {} & \pm\tfrac{1}{2}[0, \dots, N-5, N-3, N-2, N-1, N] \\ & \mp\tfrac{1}{2}[0, \dots, N-4, N-2, N-1, N+1] \\ & \pm\tfrac{1}{2}[0, \dots, N-3, N-1, N+2] \\ & \mp\tfrac{1}{2}[0, \dots, N-2, N+3] \qquad (26) \\ & \qquad\qquad \vdots \qquad\qquad . \end{aligned}$$

Hence equation (10) becomes

$$\begin{aligned} & \pm 8[0, \dots, N-1]([0, \dots, N-1][0, \dots, N-3, N, N+3] \\ & \quad -[0, \dots, N-2, N][0, \dots, N-3, N-1, N+3] \\ & \quad +[0, \dots, N-2, N+3][0, \dots, N-3, N-1, N]) \\ & \mp 8[0, \dots, N-1]([0, \dots, N-1][0, \dots, N-5, N-3, N-2, N, N+1] \\ & \quad -[0, \dots, N-2, N][0, \dots, N-5, N-3, N-2, N-1, N+1] \\ & \quad +[0, \dots, N-2, N+1][0, \dots, N-5, N-3, N-2, N-1, N]) \\ & \mp 8[0, \dots, N-2, N]([0, \dots, N-1][0, \dots, N-3, N, N+2] \\ & \quad -[0, \dots, N-2, N][0, \dots, N-3, N-1, N+2] \\ & \quad +[0, \dots, N-2, N+2][0, \dots, N-3, N-1, N]) \\ & \pm 8[0, \dots, N-2, N]([0, \dots, N-1][0, \dots, N-4, N-2, N, N+1] \\ & \quad -[0, \dots, N-2, N][0, \dots, N-4, N-2, N-1, N+1] \\ & \quad +[0, \dots, N-2, N+1][0, \dots, N-4, N-2, N-1, N]) \end{aligned}$$

$$\pm 8\{[0,\ldots,N-3,N-1,N]-[0,\ldots,N-2,N+1]\}$$
$$\times([0,\ldots,N-1][0,\ldots,N-3,N,N+1]$$
$$-[0,\ldots,N-2,N][0,\ldots,N-3,N-1,N+1]$$
$$+[0,\ldots,N-2,N+1][0,\ldots,N-3,N-1,N])=0. \qquad (27)$$

Since ( ) parts of equation (27) are nothing but Plücker relations [15], each term on the left handside of equation (27) becomes zero. □

Thus, equation (6) is proved to be a completely integrable system which must have an infinite series of invariants. With this connection we proceed to discuss the Miura transformation of equation (6). The Miura transformation in the dependent variable of equation (6) is

$$\phi_x = \psi_x^2 + \sigma\psi_{xx} + \sigma\psi_z \qquad (28)$$

with $\sigma = \pm i$. This transformation is the same as the potential KP (p-KP) equation [16]. Then we obtain the modified equation

$$\psi_{xt} + \tfrac{1}{4}\psi_{xxxz} + \sigma\psi_{xz}\psi_z + \left(\tfrac{1}{2}\psi_x + \tfrac{1}{2}\psi_{xx}\partial_x^{-1} - \tfrac{1}{4}\sigma\partial_x^{-1}\partial_z\right)\left((\psi_x^2)_z + \sigma\psi_{zz}\right) = 0. \qquad (29)$$

On the other hand, we can obtain equation (29) by modifying the L and T operators of the potential modified KdV (p-mKdV) equation with the same method as we obtained equation (6) from the p-KdV equation. The Lax operators of equation (29) are

$$L = \partial_x^2 + 2\sigma\psi_x\partial_x + \partial_z \qquad (30)$$

$$T = \partial_x^2\partial_z + \sigma\psi_z\partial_x^2 + 2\sigma\psi_x\partial_x\partial_z + \tfrac{1}{2}\partial_z^2 + \left(-2\psi_x\psi_z + \tfrac{3}{2}\sigma\psi_{xz} + \tfrac{1}{2}\partial_x^{-1}(\psi_x^2)_z - \tfrac{1}{2}\sigma\partial_x^{-1}\psi_{zz}\right)\partial_x + \partial_t \qquad (31)$$

where we have replaced $\sigma \to -\sigma$ in equation (29).

Here, concluding remarks are in order. We have obtained the exact N-soliton solution (16) and the $N\times N$ Wronskian solution (21) of equation (6) and have constructed the modified equation (29) using the Miura transformation (28). Moreover, we have constructed the Lax pair of equation (29).

In [12] we developed the construction method for the higher-dimensional integrable equations and obtained higher-dimensional equations (the p-KP equation, the p-CBS equation and equation (6)) from the p-KdV equation. This is schematically depicted in the top part of figure 1. The Miura transformation of the p-KdV equation [17] and the p-CBS equation [9, 18] is

$$\phi_x = \psi_x^2 + \sigma\psi_{xx}. \qquad (32)$$

The corresponding modified equations are called the p-mKdV equation and the modified CBS (mCBS) equation, respectively. The latter is

$$\psi_{xt} + \tfrac{1}{4}\psi_{xxxz} + \psi_x^2\psi_{xz} + \tfrac{1}{2}\partial_x^{-1}(\psi_x^2)_z = 0. \qquad (33)$$

This equation was also found by Bogoyavlenskij [19].

The Miura transformation of the p-KP equation is the same as equation (6), i.e. equation (28) [16]. In this paper, we have given the modified equation (29) of equation (6) from the Miura transformation (28) (see figure 1). Konopelchenko and Dubrovsky gave the Lax pairs of the mKdV equation and the mKP equation [2]. Using the construction method for the higher-dimensional equation, we have obtained the Lax pair of the mCBS equation (33)

$$L = \partial_x^2 + 2\sigma\psi_x\partial_x \qquad (34)$$

$$T = \partial_x^2\partial_z + \sigma\psi_z\partial_x^2 + 2\sigma\psi_x\partial_x\partial_z + \left(-2\psi_x\psi_z + \tfrac{3}{2}\sigma\psi_{xz} + \tfrac{1}{2}\partial_x^{-1}(\psi_x^2)_z\right)\partial_x + \partial_t \qquad (35)$$

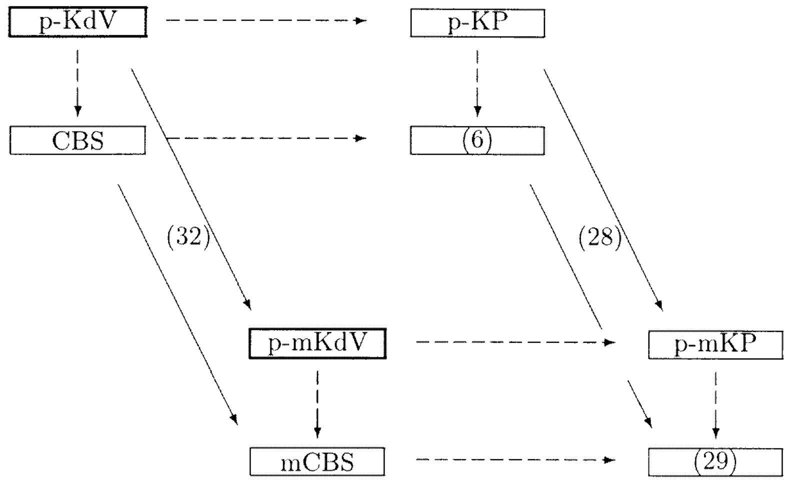

Figure 1. Scheme of extensions of the p-KdV equation and the p-mKdV equation. There are two directional routes of extensions (broken arrows): one leads us to the p-KP equation and the p-mKP equation. Another goes to the p-CBS equation and the p-mCBS equation. Equations (6) and (29) are given by unifying two routes of extensions. Four equations in the bottom part of this figure are induced by the Miura transformations (full arrows).

and the Lax pair (30), (31) of the modified equation (29) (see the bottom part of figure 1). Thus we have succeeded in constructing the Lax pairs of the higher-dimensional modified equations (the mCBS equation and equation (29)), composing the analogous ring of modified systems.

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References

[1] Zakharov V E and Shabat A B 1974 *Funct. Anal. Appl.* **8** 226
Dryuma V S 1974 *Sov. JETP Lett.* **19** 387
[2] Konopelchenko B G and Dubrovsky V G 1983 *Phys. Lett.*
[3] Calogero F 1975 *Lett. Nuovo Cimento* **14** 443
[4] Calogero F 1975 *Lett. Nuovo Cimento* **14** 537
[5] Calogero F 1976 *Nuovo Cimento* B **31** 229
[6] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* vol I (Amsterdam: North-Holland)
[7] Bogoyavlenskij O I 1990 *Math. USSR. Izv.* **34** 245
[8] Schiff J 1992 *Painlevé Transendents, Their Asymtopics and Physical Applications* (New York: Plenum) p 393
[9] Yu S, Toda K, Sasa N and Fukuyama T 1998 *J. Phys. A: Math. Gen.* **31** 3337
[10] Hietarinta J, Grammaticos B and Ramani A 1995 *Integrable Trilinear PDE's* ed V G Makhankov, A R Bishop and D D Holm (Singapore: World Scientific)
[11] Hu X and Li Y 1989 *J. Grad. Sch. USTC* **6** 8 (in Chinese)
[12] Yu S, Toda K and Fukuyama T 1998 *Preprint* solv-int/9802005
[13] Weiss J, Tabor M J and Carnevale G 1983 *J. Math. Phys.* **24** 522
[14] Grammaticos B, Ramani A and Hietarinta J 1994 *Phys. Lett.* A **190** 65
[15] Hirota R, Ohta Y and Satsuma J 1988 *Prog. Theor. Phys. Suppl.* **94** 59
[16] Konopelchenko B G 1982 *Phys. Lett.* A **92** 323
[17] Miura R 1968 *J. Math. Phys.* **9** 1202

[18] Bogoyavlenskij O I 1991 *Math. USSR. Izv* **36** 129
Bogoyavlenskij O I 1991 *Math. USSR. Izv* **102** A 15
[19] Bogoyavlenskij O I 1991 *Breaking Solitons* (Moscow: Nauka) p 69